ASYMPTOTICALLY OPTIMAL t-DESIGN CURVES ON S^3

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Abstract. A *spherical t-design curve* was defined by Ehler and Gröchenig to be a continuous, piecewise smooth, closed curve on the sphere with finitely many self-intersections whose associated line integral applied to any polynomial of degree at most t evaluates to the average of this polynomial on the sphere. These authors posed the problem of proving that there exist sequences $(\gamma_t)_{t=0}^{\infty}$ of t-design curves on S^d of asymptotically optimal length $\ell(\gamma_t) \asymp t^{d-1}$ as $t \to \infty$ and solved this problem for $d = 2$. This work solves the problem for $d = 3$ by proving that there exists a constant $\mathscr{C} > 0$ such that for any $C \geq \mathscr{C}$ and $t \in \mathbb{N}_+$, there exists a simple t-design curve on S^3 of length Ct^2.

1. Background and main results

Definition 1.1 (Definition 2.1 of Ehler and Gröchenig [6])**.** For any $d \in \mathbb{N}_+ := \{1, 2, ...\}$ and $t \in \mathbb{N} := \{0, 1, ...\}$, a continuous, piecewise smooth, closed curve $\gamma : [0, 1] \to S^d$ with finitely many self-intersections is called a *t-design curve* if

$$\frac{1}{\ell(\gamma)} \int_\gamma f := \frac{1}{\ell(\gamma)} \int_0^1 f(\gamma(s))|\gamma'(s)|\, ds = \frac{1}{|S^d|} \int_{S^d} f \, d\sigma$$

for all f in the space $P_t(S^d)$ of restrictions to S^d of polynomials on $\mathbb{R}^{d+1}$ of degree at most t, where $\ell(\gamma) := \int_0^1 |\gamma'(s)|\, ds$ is the length of γ and σ is the standard uniform measure on S^d.

The introduction of t-design curves was motivated by the numerous uses curves have found in data analysis on the sphere [6, Section 1]. These objects are a natural analogue of the well-studied [1] discrete objects *spherical t-designs*—finite point sets on spheres which provide *quadrature* or *cubature* rules, averaging degree at most t polynomials exactly—introduced by Delsarte, Goethals, and Seidel [5]. The optimal asymptotic order of size of a t-design on S^d as $t \to \infty$ was shown by Delsarte, Goethals, and Seidel—who noted that the vertices of a regular $(t+1)$-gon give a t-design on S^1 [5, Example 5.1.4]—to be t when $d = 1$ and was a long-standing open problem for $d > 1$ until Bondarenko, Radchenko, and Viazovska showed that an asymptotic lower bound on this size proven by Delsarte, Goethals, and Seidel [5, Definition 5.13] was optimal up to a constant for each $d \in \{2, 3, ...\}$, specifically proving existence of a constant $C_d > 0$ such that there exists a spherical t-design on S^d of size N for any $t \in \mathbb{N}_+$ and $N \geq C_d t^d$ [2, Theorem 1]. Ehler and Gröchenig established an asymptotic lower bound $\ell(\gamma) \geq \widetilde{c}_d t^{d-1}$ ($\widetilde{c}_d$ a constant depending on d) on the order of length of a spherical t-design curve γ on S^d for $d \in \mathbb{N}_+$ [6, Theorem 1.1] and called sequences $(\gamma_t)_{t=0}^{\infty}$ of t-design curves achieving this bound up to a constant *asymptotically optimal*. These authors used the results of Bondarenko, Radchenko, and Viazovska [2, Theorem 1] to show existence of asymptotically optimal sequences of t-design curves on S^d for $d = 2$ [6, Theorem 1.2] and posed the problem of proving existence of such sequences for $d > 2$. Theorem 1.2 solves this problem for $d = 3$.

Theorem 1.2 (Main Theorem)**.** *There exists a constant $\mathscr{C} > 0$ such that for any $C \geq \mathscr{C}$ and $t \in \mathbb{N}_+$, there exists a simple t-design curve on S^3 of length Ct^2.*

We note that applying techniques of Ehler and Gröchenig [6, Sections 4-6] to a sequence $(\gamma_t)_{t=1}^{\infty}$ of t-design curves satisfying $\ell(\gamma_t) = \mathscr{C} t^2$, one may decrease the shortest known asymptotic order of length achieved by sequences of t-design curves on S^d for $d \geq 3$ from $t^{d(d-1)/2}$ to $t^{d(d-1)/2-1}$.

To show Theorem 1.2, we combine a construction communicated by Theorem 1.3 which builds a t-design curve on S^3 from a $\lfloor t/2 \rfloor$-design curve on S^2 with the result of Ehler and Gröchenig that there exists an asymptotically optimal sequence of t-design curves on S^2.

Theorem 1.3. *Consider $t \in \mathbb{N}$ and a $\lfloor t/2 \rfloor$-design curve α on S^2. For any $\varepsilon \geq 0$, we may construct a t-design curve $\gamma_{\alpha,\varepsilon}$ on S^3 as in* (5) *which may have self-intersections for $\varepsilon = 0$ but which will be simple for $\varepsilon > 0$ of length*

$$\ell(\gamma_{\alpha,\varepsilon}) = (t+1)\sqrt{\ell(\alpha)^2 + \phi_\alpha^2} + \varepsilon \tag{1}$$

for a constant $\phi_\alpha \in (-\pi, \pi]$ chosen as in (16) *which satisfies the bound* (4)*.*

Theorem 1.4 (Theorem 1.2 of Ehler and Gröchenig [6])**.** *There exists a constant $\mathscr{A} > 0$ such that for any $t \in \mathbb{N}_+$, there exists a t-design curve on S^2 of length at most $\mathscr{A} t$.*

Theorem 1.2 directly follows from Theorems 1.3 and 1.4 by taking

$$\mathscr{C} := \sqrt{\mathscr{A}^2 + 4\pi^2},$$

as for any $t \in \mathbb{N}_+$ and α a $\lfloor t/2 \rfloor$-design curve satisfying $\ell(\alpha) \leq \mathscr{A} t$, we will have $\ell(\gamma_{\alpha,\varepsilon}) \leq \mathscr{C} t^2$ for $\varepsilon = 0$ and we may then increase the length of $\gamma_{\alpha,\varepsilon}$ as desired by increasing ε. We prove Theorem 1.3 in Section 2, then discuss examples of explicit t-design curves arising from the construction of Theorem 1.3 and results arising from natural generalizations of the theorem in Section 3. These results include explicit sequences of asymptotically optimal t-design curves on $(S^1)^d$, existence of asymptotically optimal sequences of t-design curves on $S^2 \times (S^1)^d$ and $S^3 \times (S^1)^d$, and a construction to be made fully formal in future work [11] which builds a t-design curve on S^{2n+1} from a $\lfloor t/2 \rfloor$-design curve on $\mathbb{CP}^n$ that we plan to use to give rise to improved bounds on the asymptotically smallest sequences of t-design curves known to exist on S^d for $d > 3$.

2. Building t-design curves using the Hopf map

We now give an informal overview of the proof of Theorem 1.3, present lemmas used in the proof in Subsection 2.1, and provide the formal proof in Subsection 2.2. Consider the Hopf map

$$\pi : S^3 \to S^2, \quad (a, b) \in \mathbb{C}^2 \mapsto (|a|^2 - |b|^2, 2a\bar{b}) \in \mathbb{R} \times \mathbb{C}, \tag{2}$$

which gives rise to a principal S^1-bundle with fibers

$$\pi^{-1}(w) = \{\omega z \mid z \in S^1 \subset \mathbb{C}\} \cong S^1 \tag{3}$$

for $w \in S^2$ and any $\omega \in \pi^{-1}(w)$. To build the t-design curve $\gamma_{\alpha,\varepsilon}$ on S^3 discussed in Theorem 1.3 from a $\lfloor t/2 \rfloor$-design curve α on S^2, we first use Lemma 2.1 to lift α to a curve β_α on S^3 satisfying $\pi \circ \beta_\alpha = \alpha$ whose derivative $\beta'_\alpha(s)$ is always orthogonal to the tangent space of the fiber $\pi^{-1}(\pi(\beta_\alpha(s)))$. We then rotate β_α fiberwise such that the concatenation $\gamma_{\alpha,\varepsilon}$ of $t+1$ appropriately rotated copies of the resulting curve will be continuous, piecewise smooth, closed, and, if $\varepsilon > 0$, to remove any self-intersections and lengthen the curve by ε. We note that the lengthening of the curve done to ensure that $\gamma_{\alpha,\varepsilon}$ will be closed results in the constant ϕ_α in the formula (1) for the length of $\gamma_{\alpha,\varepsilon}$. We may observe from our choice (16) of ϕ_α that ϕ_α may be expressed as a function of the area enclosed by α using the Gauss-Bonnet theorem [7, 3]. Additionally, denoting by G_{t+1} the set of generators of the cyclic group of order $t+1$ (natural numbers less than and coprime to $t+1$), it is straightforward to see that

$$|\phi_\alpha| \leq \frac{\pi}{t+1} \max_{g_1, g_2 \in G_{t+1}} \min(|g_1 - g_2|, t+1 - |g_1 - g_2|) \tag{4}$$

for $t > 2$, which gives rise to the bound $|\phi_\alpha| \leq \frac{2\pi}{t+1}$ when $t+1$ is prime.

The construction described above will arrange for all $s \in [0, 1]$ that

$$(\pi \circ \gamma_{\alpha,\varepsilon})(s) = \alpha((t+1)s - \lfloor (t+1)s \rfloor)$$

and that

$$(\gamma_{\alpha,\varepsilon}([0, 1])) \cap \pi^{-1}(\pi(\gamma_{\alpha,\varepsilon}(s)))$$

is the vertices of a regular $(t+1)$-gon on $\pi^{-1}(\pi(\gamma_{\alpha,\varepsilon}(s))) \cong S^1$, which we see from Lemma 2.4—the result of an example [5, Example 5.1.4] of Delsarte, Goethals, and Seidel—is a t-design on the fiber. We show that such a curve will be a t-design curve on S^3 using a method analogous to those used first by König [9, Corollary 1] and Kuperberg [10, Theorem 4.1] to relate t-designs on spheres to $\lfloor t/2 \rfloor$-designs on quotient complex projective spaces, later by Okuda [13, Theorem 1.1] (who was inspired by work of Cohn, Conway, Elkies, and Kumar [4, Section 4] on the subject) to relate t-designs on S^3 to $\lfloor t/2 \rfloor$-designs on S^2, and most recently by the present author [12, Theorem 1.1] to relate t-designs on spheres to $\lfloor t/2 \rfloor$-designs on quotient real, complex, quaternionic, or octonionic projective spaces or spheres. To this end, we will present Lemmas 2.2 and 2.3, which were used by Okuda in formalizing their result [13, Theorem 1.1].

We may observe from this outline of the construction of $\gamma_{\alpha,\varepsilon}$ and the presentation

$$\pi^{-1}(\xi,\eta) = \begin{cases} \left\{ \frac{1}{\sqrt{2}} \left(\zeta\sqrt{1+\xi}, \frac{\overline{\eta}\zeta}{\sqrt{1+\xi}} \right) \,\Big|\, \zeta \in S^1 \right\}, & \xi \neq -1 \\ \{(0,\zeta) \mid \zeta \in S^1\}, & \xi = -1 \end{cases}$$

of preimages of π that

$$\gamma_{\alpha,\varepsilon}(s) = \begin{cases} \frac{1}{\sqrt{2}} \left(\sqrt{1+\alpha_1(r)}, \frac{\alpha_2(r) - i\alpha_3(r)}{\sqrt{1+\alpha_1(r)}} \right) e^{2\pi i s + i\theta(r)}, & \alpha_1(r) \neq -1 \\ (0,1)\, e^{2\pi i s + i\theta(r)}, & \alpha_1(r) = -1 \end{cases} \tag{5}$$

for $r := (t+1)s - \lfloor (t+1)s \rfloor$ and a piecewise smooth function $\theta : [0,1] \to \mathbb{R}$ satisfying $\theta(0) - \theta(1) \in 2\pi\mathbb{Z}$ which will be continuous at all points $r \in [0,1]$ satisfying $\alpha_1(r) \neq -1$.

2.1. **Curves and polynomials related through the Hopf map.** We now present lemmas used in the proof of Theorem 1.3. Lemma 2.1 discusses how to lift a curve on S^2 to a curve on S^3 whose derivative is orthogonal to the tangent space of each Hopf fiber it passes through, Lemma 2.2 discusses how an integrable function on S^3 can be averaged by averaging the function on S^2 which averages it on each Hopf fiber, Lemma 2.3 relates polynomials on S^3 to those on S^2 and on Hopf fibers, and Lemma 2.4 discusses how the vertices of a regular $(t+1)$-gon on a Hopf fiber give a t-design on the fiber. We first show Lemma 2.1. To this end, observe that since the Hopf map (2) constitutes a fiber bundle, we have the decomposition

$$T_\omega S^3 \cong T_\omega(\pi^{-1}(w)) \oplus T_w S^2 \tag{6}$$

for $\omega \in S^3$ and $w := \pi(\omega) \in S^2$. There is a natural isometric inclusion

$$\iota_\omega : T_w S^2 \hookrightarrow T_\omega S^3$$

satisfying

$$d\pi_\omega \circ \iota_\omega = \mathrm{id}_{T_w S^2}, \tag{7}$$

where $\mathrm{id}_{T_w S^2}$ is the identity map on $T_w S^2$.

Lemma 2.1. *Take a continuous, piecewise smooth curve* $\alpha : [0,1] \to S^2$ *with finitely many self-intersections. We may construct a continuous, piecewise smooth curve* $\beta_\alpha : [0,1] \to S^3$ *with finitely many self-intersections satisfying*

$$\beta'_\alpha(s) = \iota_{\beta_\alpha(s)}(\alpha'(s)) \tag{8}$$

for all $s \in [0,1]$ *at which* α *is smooth and*

$$\pi \circ \beta_\alpha = \alpha. \tag{9}$$

Proof. Consider α as in the lemma and a partition $0 = s_0 < \cdots < s_m = 1$ of $[0,1]$ such that α is smooth and has no self-intersections on each interval $I_j := [s_j, s_{j+1}]$. Fixing $j \in \{0, ..., m-1\}$, the restriction $\alpha_j := \alpha|_{I_j}$ is then a diffeomorphism onto its image $\alpha(I_j)$, so $\pi^{-1}(\alpha(I_j))$ is a smooth submanifold (with boundary) of S^3. We define a vector field V_j on $\pi^{-1}(\alpha(I_j)) \ni \omega$ by

$$(V_j)_\omega = \iota_\omega(\alpha'(\alpha_j^{-1}(\pi(\omega)))).$$

The method of successive approximations allows us to construct the unique smooth flow

$$\beta_j : \pi^{-1}(\alpha(s_j)) \times I_j \to S^3$$

satisfying

$$\begin{aligned} \beta_j(\omega, s_j) &= \omega, \\ \frac{\partial \beta_j}{\partial s}(\omega, s) &= (V_j)_{\beta_j(\omega,s)} \end{aligned} \tag{10}$$

for $\omega \in \pi^{-1}(\alpha(s_j))$ and $s \in I_j$. For such ω, we have

$$(\pi \circ \beta_j)(\omega, s_j) = \pi(\omega) = \alpha(s_j)$$

and (7) shows that for $s \in I_j$ satisfying $(\pi \circ \beta_j)(\omega, s) = \alpha(s)$, we have

$$\begin{aligned} \frac{\partial(\pi \circ \beta_j)}{\partial s}(\omega, s) &= \left(d\pi_{\beta_j(\omega,s)} \circ \frac{\partial \beta_j}{\partial s} \right)(\omega, s) \\ &= d\pi_{\beta_j(\omega,s)}((V_j)_{\beta_j(\omega,s)}) \\ &= \alpha'(\alpha_j^{-1}((\pi \circ \beta_j)(\omega, s))) \\ &= \alpha'(s). \end{aligned}$$

So, we must have

$$(\pi \circ \beta_j)(\omega, s) = \alpha(s) \tag{11}$$

for $\omega \in \pi^{-1}(\alpha(s_j))$ and $s \in I_j$, showing that β_j has no self-intersections since α_j has none.

Fix $\omega \in \pi^{-1}(\alpha(0))$. We now consider the curve $\beta_\alpha : [0,1] \to S^3$ which arises from flowing along each β_j successively starting at ω, so we have $\beta_\alpha(0) = \omega$ and $\beta_\alpha(s) = \beta_j(\beta_\alpha(s_j), s)$ for $j \in \{0, ..., m-1\}$ and $s \in I_j$. By this construction, we see since β_j is smooth and has no self-intersections for each j that β_α is continuous and piecewise smooth with finitely many self-intersections. Moreover, we directly see from (10) and (11) that (8) and (9) hold. $\square$

Lemmas 2.2 and 2.3 are concerned with properties of the operator I_π which takes $f \in L^1(S^3)$ to the function

$$(I_\pi f)(w) = \frac{1}{2\pi} \int_{\pi^{-1}(w)} f \, d\sigma \tag{12}$$

on S^2 which gives the average of f on each Hopf fiber (where σ is the standard uniform measure on $\pi^{-1}(w) \cong S^1$) and the left multiplication by a base point ω isomorphism

$$\zeta_\omega : S^1 \to \pi^{-1}(w), \quad \zeta \mapsto \omega\zeta \tag{13}$$

we define for $\omega \in S^3$ and $w := \pi(\omega) \in S^2$. We do not prove these lemmas, but note that proofs can be found in work of Okuda [13, Lemmas 4.2-4.3] or of the present author [12, Lemmas 2.2-2.3, 2.5].

Lemma 2.2 (Lemma 4.2 of Okuda [13])**.** *For $f \in L^1(S^3)$, we have*

$$\frac{1}{|S^2|} \int_{S^2} I_\pi f \, d\sigma = \frac{1}{|S^3|} \int_{S^3} f \, d\sigma.$$

Lemma 2.3 (Lemma 4.3 of Okuda [13])**.** *We have*

$$I_\pi(P_t(S^3)) = P_{\lfloor t/2 \rfloor}(S^2), \tag{14}$$

$$\zeta_\omega^*(P_t(S^3)) = P_t(S^1) \quad \text{for any} \quad \omega \in S^3. \tag{15}$$

Lemma 2.4 follows from (15) paired with the fact [5, Example 5.1.4] noted by Delsarte, Goethals, and Seidel that the vertices of a regular $(t+1)$-gon give a t-design on S^1.

Lemma 2.4 (Example 5.1.4 of Delsarte, Goethals, and Seidel [5])**.** *For any $\omega \in S^3$ and $f \in P_t(S^3)$, we have*

$$\frac{1}{t+1} \sum_{j=0}^{t} f(\omega e^{2\pi i j/(t+1)}) = (I_\pi f)(\pi(\omega)).$$

2.2. **Formal treatment of the construction.** We now prove Theorem 1.3, applying Lemma 2.1 to assist in constructing the desired curve $\gamma_{\alpha,\varepsilon}$ and using Lemmas 2.2, 2.3, and 2.4 to show that $\gamma_{\alpha,\varepsilon}$ is a t-design curve.

Proof of Theorem 1.3. Consider α as in the theorem statement, which we reparameterize so its derivative has constant norm $|\alpha'| = \ell(\alpha)$. Consider β_α as in Lemma 2.1 alongside the generators G_{t+1} of the cyclic group of order $t+1$. Since α is a closed curve, (9) shows that $\beta_\alpha(0)$ lies in the image $\pi^{-1}(\pi(\beta_\alpha(1)))$ of the isomorphism $\zeta_{\beta_\alpha(1)}$ as in (13). Therefore, we may pick $g \in G_{t+1}$ minimizing $|\phi_\alpha|$ for $\phi_\alpha \in (-\pi, \pi]$ defined by

$$e^{i\phi_\alpha} = \zeta_{\beta_\alpha(1)}^{-1}(\beta_\alpha(0)) e^{2\pi i g/(t+1)} \in S^1 \subset \mathbb{C}. \tag{16}$$

With ε as in the theorem statement, fix

$$\phi_\varepsilon := \sqrt{\phi_\alpha^2 + \frac{2\varepsilon\sqrt{\ell(\alpha)^2+\phi_\alpha^2}}{t+1} + \frac{\varepsilon^2}{(t+1)^2}},$$

so we have

$$\sqrt{\ell(\alpha)^2+\phi_\varepsilon^2} = \sqrt{\ell(\alpha)^2+\phi_\alpha^2} + \frac{\varepsilon}{t+1}. \tag{17}$$

We may consider the partition $0 = s_0 < \cdots < s_m = 1$ of $[0,1]$ arising from the union of the set of self-intersection points of α with $\{0,1\}$. We define

$$r_{j,\delta} := \frac{1}{2}(s_{j-1} + s_j + (s_j - s_{j-1})\phi_\alpha/\phi_\varepsilon) + \delta \quad \textit{for} \quad j \in \{1, ..., m-1\},$$
$$r_{m,\delta} := \frac{1}{2}(s_{m-1} + 1 + (1 - s_{m-1})\phi_\alpha/\phi_\varepsilon) - (m-1)\delta$$

for some fixed $\delta \in [0, \Delta]$, where we have

$$\Delta = \min\left\{s_1 - r_{1,0}, ..., s_{m-1} - r_{m-1,0}, \frac{r_{m,0} - s_{m-1}}{m-1}\right\}$$

so that $r_{j,\delta} \in [s_{j-1}, s_j]$ for all j. Setting

$$I_{+,\delta} := \bigcup_{j=1}^{m}(s_{j-1}, r_{j,\delta}), \quad I_{-,\delta} := \bigcup_{j=1}^{m}(r_{j,\delta}, s_j),$$

we can directly compute that

$$|[0,s_j] \cap I_{+,\delta}| - |[0,s_j] \cap I_{-,\delta}| = s_j\phi_\alpha/\phi_\varepsilon + 2j\delta \quad \textit{for} \quad j \in \{1, ..., m-1\},$$
$$|I_{+,\delta}| - |I_{-,\delta}| = \phi_\alpha/\phi_\varepsilon.$$

Thus, considering the unique continuous function $\theta_\delta : [0,1] \to \mathbb{R}$ defined by

$$\theta_\delta(0) = 0, \quad \theta_\delta'(s) = \pm\phi_\varepsilon \quad \textit{for} \quad s \in I_{\pm,\delta},$$

we have that

$$\begin{aligned} \theta_\delta(s_j) &= s_j\phi_\alpha + 2j\delta\phi_\varepsilon \quad \textit{for} \quad j \in \{0, ..., m-1\}, \\ \theta_\delta(1) &= (|I_{+,\delta}| - |I_{-,\delta}|)\phi_\varepsilon = \phi_\alpha. \end{aligned} \tag{18}$$

Defining functions

$$q(s) = \frac{\lfloor (t+1)s \rfloor}{t+1}, \quad r(s) = (t+1)(s - q(s))$$

for $s \in [0,1]$, we consider the continuous, piecewise smooth curve

$$\widetilde{\gamma}_{\alpha,\delta} := (\beta_\alpha \circ r)e^{i(\theta_\delta \circ r + 2\pi g q)} : [0,1] \to S^3.$$

We may directly see that

$$\widetilde{\gamma}_{\alpha,\delta}(1) = \beta_\alpha(0) = \widetilde{\gamma}_{\alpha,\delta}(0),$$

so $\widetilde{\gamma}_{\alpha,\delta}$ is a closed curve. Observe from (9) that

$$\pi \circ \widetilde{\gamma}_{\alpha,\delta} = \pi \circ \beta_\alpha \circ r = \alpha \circ r,$$

so $\widetilde{\gamma}_{\alpha,\delta}$ may only have a self-intersection at a point $s \in [0,1]$ if there exists $\tilde{s} \in [0,1]$ such that $r(s) = r(\tilde{s})$ or if $r(s)$ is a self-intersection point of α. If $r(s) = r(\tilde{s})$, we have $\tilde{s} = s + k/(t+1)$ for some $k \in \{-t, ..., t\}$, so

$$\widetilde{\gamma}_{\alpha,\delta}(\tilde{s}) = \beta_\alpha(r(s))e^{i(\theta_\delta(r(s))+2\pi g(q(s)+k/(t+1)))} = \widetilde{\gamma}_{\alpha,\delta}(s)e^{2\pi igk/(t+1)} \neq \widetilde{\gamma}_{\alpha,\delta}(s)$$

since g is a generator of the cyclic group of $t+1$ elements and thus gk will not be an integer multiple of $t+1$. Therefore, $\widetilde{\gamma}_{\alpha,\delta}$ may only have self-intersections in

$$r^{-1}(\{s_j\}_{j=0}^m) = \left\{ s_{j,k} := \frac{s_j + k}{t+1} \;\middle|\; j \in \{0, ..., m\},\ k \in \{0, ..., t\} \right\}.$$

For any $k \in \{0, ..., t\}$, (18) and (16) show that

$$\begin{aligned} \widetilde{\gamma}_{\alpha,\delta}(s_{j,k}) &= \beta_\alpha(s_j)e^{i(s_j\phi_\alpha+2j\delta\phi_\varepsilon+2\pi gk/(t+1))} \quad \textit{for} \quad j \in \{0, ..., m-1\}, \\ \widetilde{\gamma}_{\alpha,\delta}(s_{m,k}) &= \beta_\alpha(1)e^{i(\phi_\alpha+2\pi gk/(t+1))} = \beta_\alpha(0)e^{2\pi ig(k+1)/(t+1)}, \end{aligned}$$

so we can see that there will only be finitely many $\delta \in [0,\Delta]$ such that $\widetilde{\gamma}_{\alpha,\delta}$ will have any self-intersections. We take $\gamma_{\alpha,\varepsilon} := \widetilde{\gamma}_{\alpha,0}$ when $\varepsilon = 0$. Otherwise, we have $\Delta > 0$, so we may pick $\delta \in [0,\Delta]$ such that $\widetilde{\gamma}_{\alpha,\delta}$ has no self-intersections and label $\gamma_{\alpha,\varepsilon} := \widetilde{\gamma}_{\alpha,\delta}$.

Combining (8), the fact that ι_ω is an isometry for all $\omega \in S^3$, and our assumption that $|\alpha'|$ is constant, we see that $|\beta_\alpha'| = |\alpha'| = \ell(\alpha)$, so the decomposition (6) shows that

$$|\gamma_{\alpha,\varepsilon}'|^2 = |(\beta_\alpha \circ r)'|^2 + |(\theta_\delta \circ r)'|^2 = (t+1)^2(\ell(\alpha)^2 + \phi_\varepsilon^2) \tag{19}$$

at all points where $\gamma_{\alpha,\varepsilon}$ is smooth. Thus, we have

$$\ell(\gamma_{\alpha,\varepsilon}) = (t+1)\sqrt{\ell(\alpha)^2 + \phi_\varepsilon^2}$$

and (17) then shows that (1) is satisfied.

To complete the proof, we need only show that for any $f \in P_t(S^3)$,

$$\frac{1}{\ell(\gamma_{\alpha,\varepsilon})}\int_{\gamma_{\alpha,\varepsilon}} f = \frac{1}{|S^3|}\int_{S^3} f\,d\sigma. \tag{20}$$

Picking such f, as (19) shows that $|\gamma_{\alpha,\varepsilon}'|$ is constant and thus equals $\ell(\gamma_{\alpha,\varepsilon})$, we have

$$\frac{1}{\ell(\gamma_{\alpha,\varepsilon})}\int_{\gamma_{\alpha,\varepsilon}} f = \frac{1}{\ell(\gamma_{\alpha,\varepsilon})}\int_0^1 f(\gamma_{\alpha,\varepsilon}(s))\ell(\gamma_{\alpha,\varepsilon})\,ds = \int_0^1 f(\gamma_{\alpha,\varepsilon}(s))\,ds. \tag{21}$$

With I_π as in (12), we then see from Lemma 2.4 and since g is a generator of the cyclic group of order $t+1$ that

$$\frac{1}{t+1}\sum_{k=0}^{t} f(\omega e^{2\pi igk/(t+1)}) = (I_\pi f)(\pi(\omega)),$$

so applying a change of variables $s \mapsto \frac{s}{t+1}$, (9), and that $|\alpha'| = \ell(\alpha)$, we have

$$\begin{aligned}
\int_0^1 f(\gamma_{\alpha,\varepsilon}(s))\,ds &= \sum_{k=0}^{t} \int_{k/(t+1)}^{(k+1)/(t+1)} f(\gamma_{\alpha,\varepsilon}(s))\,ds \\
&= \int_0^1 \frac{1}{t+1} \sum_{k=0}^{t} f(\beta_\alpha(s) e^{i(\theta_\delta(s)+2\pi gk)})\,ds \\
&= \int_0^1 (I_\pi f)((\pi \circ \beta_\alpha)(s))\,ds \qquad (22)\\
&= \int_0^1 (I_\pi f)(\alpha(s))\,ds \\
&= \frac{1}{\ell(\alpha)} \int_\alpha I_\pi f.
\end{aligned}$$

We see from (14) in Lemma 2.3 that $I_\pi f \in P_{\lfloor t/2 \rfloor}(S^2)$, so since α is a $\lfloor t/2 \rfloor$-design, (21) and (22) combine to show that

$$\frac{1}{\ell(\gamma_{\alpha,\varepsilon})} \int_{\gamma_{\alpha,\varepsilon}} f = \frac{1}{|S^2|} \int_{S^2} I_\pi f\, d\sigma.$$

Lemma 2.2 then shows that (20) is satisfied. □

3. Explicit t-design curves and generalizations

We now discuss examples of explicit t-design curves arising from the construction of Theorem 1.3. To this end, observe that the curve $\alpha(s) = (0, \cos(2\pi s), \sin(2\pi s))$ which traces around the equator of S^2 is a 1-design curve on S^2. Taking $t \in \{2, 3\}$ and building $\gamma_{\alpha,0}$ as in Theorem 1.3, we produce a smooth, simple t-design curve

$$[0,1] \to S^3, \quad s \mapsto \frac{1}{\sqrt{2}}(\cos(2\pi s),\, \sin(2\pi s),\, \cos(2\pi ts)\,, -\sin(2\pi ts)) \qquad (23)$$

on S^3 which we may observe has length $\pi\sqrt{2t^2+2}$. This curve for $t = 3$ is pictured in Figure 1.

We can similarly use the construction of Theorem 1.3 to produce smooth, simple t-design curves on S^3 for $t \in \{4, 5, 6, 7\}$ from the smooth, simple 2- and 3-design curves discovered by Ehler and Gröchenig [6, Example 3.3]. For $t \in \{0, ..., 27\}$, we can produce further explicit examples of $\lfloor t/2 \rfloor$-design curves on S^2 by applying the construction of Ehler and Gröchenig [6, Section 5] which can be used to build a $\lfloor t/2 \rfloor$-design curve on S^2 from a $\lfloor t/2 \rfloor$-design on S^2 to the $\lfloor t/2 \rfloor$-designs on S^2 compiled by Hardin and Sloane [8, Table I], and the construction of Theorem 1.3 applied to these curves gives numerous additional examples of simple t-design curves on S^3.

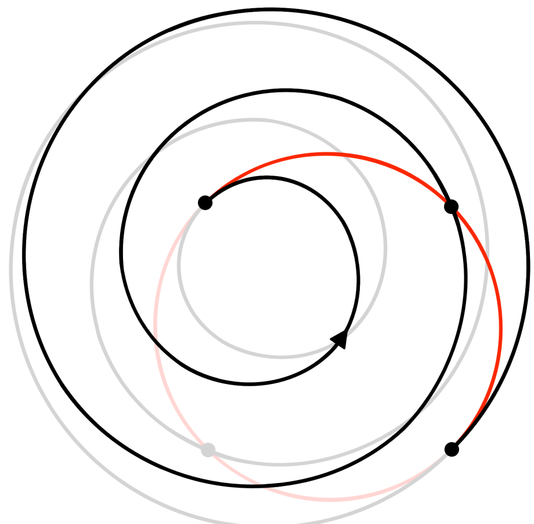

Figure 1. The 3-design curve $\gamma_{\alpha,0}$ on $S^3 \cong \mathbb{R}^3 \cup \{\infty\}$ as in (23) for α the curve which traces around the equator in S^2. $\gamma_{\alpha,0}$ lies on the Clifford torus that is the preimage of the equator of S^2 under the Hopf map; the segment of the curve on the same side of this torus as the observer is shown in black and the further segment is shown in gray. The preimage $\pi^{-1}(p)$ of a point on the equator of S^2 under the Hopf map is shown in red. Note the intersection of $\pi^{-1}(p)$ with $\gamma_{\alpha,0}([0,1])$ is the vertices of a square, a 3-design on $\pi^{-1}(p)$.

We now discuss results related to Theorem 1.3 concerning t-design curves on spaces other than spheres. Define, for any $t \in \mathbb{N}$ and $n \in \mathbb{N}_+$, a t-design curve on a measure space $(\Sigma \subset \mathbb{R}^n, \mu)$ to be a continuous, piecewise smooth, closed curve $\gamma : [0,1] \to \Sigma$ with finitely many self-intersections whose associated line integral exactly averages the restrictions of elements of $P_t(\mathbb{R}^n)$ to Σ as in Definition 1.1. Consider a t-design curve α on such (Σ, μ). The curve

$$[0,1] \ni s \to (\alpha((t+1)s - \lfloor (t+1)s \rfloor), e^{2\pi i s}) \in \Sigma \times S^1$$

can be shown exactly as in the proof of Theorem 1.3 (substituting for the Hopf map the projection $\Sigma \times S^1 \to \Sigma$, which similarly gives rise to a principal S^1-bundle) to be a t-design curve on $(\Sigma \times S^1 \subset \mathbb{R}^{n+2}, \mu \times \sigma)$. As the curve

$$[0,1] \ni s \mapsto e^{2\pi i s} \in S^1$$

is trivially a t-design curve on S^1 for any $t \in \mathbb{N}$, we see that

$$\gamma_t : [0,1] \ni s \to (e^{2\pi i (t+1)s}, e^{2\pi i s}) \in S^1 \times S^1$$

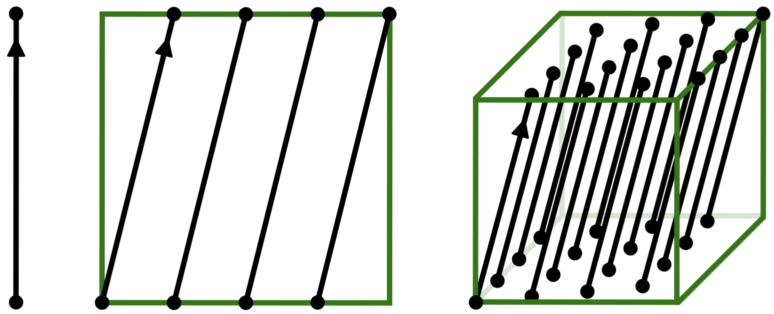

FIGURE 2. 3-design curves on S^1, $S^1 \times S^1$, and $S^1 \times S^1 \times S^1$ constructed as in (24). For a $\lfloor t/2 \rfloor$-design curve α on S^2, the curve $\gamma_{\alpha,0}$ as in Theorem 1.3 can be constructed by mapping such a t-design curve on $S^1 \times S^1$ onto the immersed torus $\pi^{-1}(\alpha([0,1]))$.

is a t-design curve on the Clifford torus $S^1 \times S^1$ of length

$$\ell(\gamma_t) = 2\pi\sqrt{(t+1)^2+1} \leq 2\pi\sqrt{5}t.$$

We then see that for any $d \in \mathbb{N}_+$,

$$\gamma_{t,d} : [0,1] \ni s \to (e^{2\pi i(t+1)^{d-1}s}, e^{2\pi i(t+1)^{d-2}s}, \ldots, e^{2\pi i s}) \in (S^1)^d \tag{24}$$

is a t-design curve on $(S^1)^d$ of length

$$\ell(\gamma_{t,d}) = 2\pi\sqrt{\sum_{j=0}^{d-1}(t+1)^{2j}} \leq \mathscr{C}_{(S^1)^d} t^{d-1}$$

for the constant

$$\mathscr{C}_{(S^1)^d} := 2\pi\sqrt{\sum_{j=0}^{d-1} 4^j}.$$

Noting that the curves (24) may be lengthened by any arbitrary amount as in the proof of Theorem 1.3 when $d > 1$, for any such d, any $C \geq \mathscr{C}_{(S^1)^d}$, and $t \in \mathbb{N}_+$, we may construct a smooth, simple t-design curve on $(S^1)^d$ of length Ct^{d-1}. We may similarly show using Theorem 1.4 when $n = 2$ and Theorem 1.2 when $n = 3$ that there exists a constant $\mathscr{C}_{S^n \times (S^1)^d}$ such that for any $C \geq \mathscr{C}_{S^n \times (S^1)^d}$ and $t \in \mathbb{N}_+$, there exists a (simple, for $n = 3$) t-design curve on $S^n \times (S^1)^d$ of length Ct^{n-1+d}. A proof analogous to one of Ehler and Gröchenig [6, Theorem 1.1] can be used to show that all of these curves achieve the minimal possible asymptotic order of length among such curves.

For $\mathbb{K}$ the reals or complex numbers, $k := \dim_{\mathbb{R}} \mathbb{K} - 1$, and $n \in \mathbb{N}_+$, we consider the $\mathbb{K}$-projective space

$$\mathbb{KP}^n = \{[\omega] \mid \omega \in S^{(k+1)(n+1)-1} \subset \mathbb{K}^{n+1}\} \quad ([\omega] := \{\omega\zeta \mid \zeta \in S^k \subset \mathbb{K}\}),$$

which we realize as a subset of $\mathbb{K}^{(n+1)^2} \cong \mathbb{R}^{(k+1)(n+1)^2}$ via the embedding taking an element $[\omega] \in \mathbb{KP}^n$ to the matrix $\omega\omega^*$ which projects $\mathbb{K}^{n+1}$ to the span over $\mathbb{R}$ of $[\omega]$. This provides us with a notion of t-design curves on $\mathbb{KP}^n$. We may then use t-design curves on S^2 and S^3 with antipodally symmetric image (which can be built on S^2 using the construction described by Ehler and Gröchenig [6, Section 5] and on S^3 using the construction of Theorem 1.3) to build $\lfloor t/2 \rfloor$-design curves on $\mathbb{RP}^2$ and $\mathbb{RP}^3$ respectively with half the lengths of the original curves. For any $d \in \mathbb{N}$, we then get existence results for t-design curves analogous to those for $S^2 \times (S^1)^d$ and $S^3 \times (S^1)^d$ on $\mathbb{RP}^2 \times (S^1)^d$ and $\mathbb{RP}^3 \times (S^1)^d$. For any $n \in \mathbb{N}$, we also plan [11] to provide an existence result for t-design curves on $\mathbb{CP}^n$ achieving an asymptotic order of length such that we can prove existence of t-design curves on S^{2n+1} for $n > 1$ asymptotically shorter than the current asymptotically shortest such curves shown to exist [6, Theorem 1.3] by combining this existence result with a generalization of Theorem 1.3 which uses the complex projective map $S^{2n+1} \ni \omega \mapsto [\omega] \in \mathbb{CP}^n$ (which can be identified with the Hopf map when $n = 1$ via the association $\mathbb{CP}^1 \cong S^2$) to build a simple t-design curve on S^{2n+1} of length approximately $(t+1)\ell(\alpha)$ from a $\lfloor t/2 \rfloor$-design curve α on $\mathbb{CP}^n$.

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